BESSEL INTEGRALS, PERIODS and ZETA NUMBERS

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Abstract: we present a summary of recent and older results on Bessel integrals and their relation with zeta numbers.

1 INTRODUCTION

We focus on Bessel integrals

$$\int_0^\infty \mathrm{d}u\; u^{n+1} K_0(u)^\kappa \tag{1}$$

with $n \geq 0$ and $\kappa \geq 1$ integers and on their relation with zeta numbers. We present results from [1, 2, 3, 4] and review [5] for a self contained presentation. Continous fractions for $\zeta(3)$, $\zeta(2)$ and $\psi_1(1/3)-\psi_1(2/3)$ are presented, some of them related to (1), others sustained by numerical PSLQ evidence [6]. All these continuous fractions are intimately connected to Apéry's continous fractions for $\zeta(3)$ and $\zeta(2)$ irrationality demonstrations [7]. Attempts to integrate Bessel integrals starting from a multi-integral representation on a finite domain are given. In the process Bessel integrals are shown to be periods[1]. Finally a possible way to address the irrationality of $\zeta(5)$ is proposed.

2 REVIEW OF [5]

2.1 Quantum Mechanics and Bessel Integrals

The random magnetic impurity model [1] describes a quantum particle in a plane coupled to a random distribution of Aharonov-Bohm fluxes perpendicular to the plane. It was introduced having in mind the Integer Quantum

[1] Periods are defined in [8] as "values of absolutely convergent integrals of rational functions with rational coefficients over domains in R^n given by polynomial inequalities with rational coefficients".

Hall effect. A perturbative expansion of the partition function of the model in the coupling constant α (the flux expressed in unit of the quantum of flux) for 2 impurities, i.e. at second order in the impurity density ρ, lead us to consider Feynmann diagrams at order $\rho^2\alpha^4$ [1] and $\rho^2\alpha^6$ [3].

On the one hand, these Feymann diagrams, which reduce after momenta integrations to multiple integrals on intermediate temperatures, were shown to rewrite in terms of simple [2] and double nested integrals [3] on products of modified Bessel functions

$$\begin{aligned}
I_{\rho^2\alpha^4} =&\int_0^\infty u\, K_0(u)^2(uK_1(u))^2\mathrm{d}u\\
I_{\rho^2\alpha^6} =&8\int_0^\infty \mathrm{d}u\, u\, K_0(u)^2(uK_1(u))^2\int_0^u \mathrm{d}x\, xK_1(x)I_1(x)K_0(x)^2\\
&-4\int_0^\infty \mathrm{d}u\, uK_0(u)(uK_1(u))\,(uK_1(u)I_0(u)-uI_1(u)K_0(u))\int_u^\infty \mathrm{d}x\, x\, K_0(x)^2K_1(x)^2\\
&+\int_0^\infty u\, K_0(u)^4(uK_1(u))^2\mathrm{d}u
\end{aligned} \tag{2}$$

On the other hand, one could show [2] by direct integration

$$\int_0^\infty uK_0(u)^4\mathrm{d}u = \frac{2^3-1}{8}\zeta(3) \tag{3}$$

and by integration by part that

$$\int_0^\infty u\, K_0(u)^2(uK_1(u))^2\mathrm{d}u \tag{4}$$

is a linear combination with rational cefficients[2] of 1 and $\int_0^\infty uK_0(u)^4\mathrm{d}u$, i.e. of 1 and $(2^3-1)\zeta(3)$. Similarly one obtained [3, 4] by direct integration[3]

$$\int_0^\infty \mathrm{d}u\, uI_0(u)K_0(u)^3 = \frac{2^2-1}{8}\zeta(2) \tag{5}$$

and by integration by part that

$$\int_0^\infty \mathrm{d}u\, uK_0(u)(uK_1(u))^2I_0(u) \tag{6}$$

and

$$\int_0^\infty \mathrm{d}u\, uK_0(u)^2uK_1(u)uI_1(u) \tag{7}$$

[2] Linear combination with rational coefficients means here that there exist three positive or negative integers a, b and c such that

$$a\int_0^\infty u\, K_0(u)^2(uK_1(u))^2\mathrm{d}u + b\int_0^\infty uK_0(u)^4\mathrm{d}u + c = 0$$

[3] See Appendix A for the derivations of (3) and (5).

are linear combinations of 1 and $\int_0^\infty \mathrm{d}u\, u I_0(u) K_0(u)^3$, i.e. of 1 and $(2^2 - 1)\zeta(2)$. Likewise, by integration by part [5]

$$\int_0^\infty u\, K_0(u)^4 (u K_1(u))^2\, \mathrm{d}u = \frac{2}{15} \int_0^\infty u\, K_0(u)^6\, \mathrm{d}u - \frac{1}{5} \int_0^\infty u^3\, K_0(u)^6\, \mathrm{d}u \tag{8}$$

It was then natural to argue [5] that (2) might also rewrite as a linear combination with rational coefficients of simple integrals on product of Bessel functions of weight 6 -defined as the total power of Bessel functions- and of zeta numbers of weigth $6 - 1 = 5$, like $\zeta(5)$, or below. The "counting rule" inferred from (3, 4, 5, 6, 7, 8) is that an integration $\int_0^\infty \mathrm{d}u\ u^n(\quad)$ with n odd diminishes the Bessel weight by one, so $\int_0^\infty u\, K_0(u)^4 (u K_1(u))^2$ is like $\zeta(5)$, that a I-Bessel function has no weight, so $\int_0^\infty \mathrm{d}u\, u I_0(u) K_0(u)^3$ is like $\zeta(2)$ and $\int_0^\infty \mathrm{d}u\, u^3\, K_0(u)^2 K_1(u)^2 \int_0^u \mathrm{d}x\, x K_1(x) I_1(x) K_0(x)^2$ like $\zeta(3)\zeta(2)$ i.e. like $\zeta(5)$, that in turn is like $\int_0^\infty u\, K_0(u)^6$ or $\int_0^\infty u^3\, K_0(u)^6$ -why only odd powers u and u^3 appear here and no higher power will become clear later. Indeed a numerical PSLQ [6] search gave

$$I_{\rho^2\alpha^6} =_{\mathrm{PSLQ}} \frac{1}{30} \int_0^\infty u\, K_0(u)^6\, \mathrm{d}u + \frac{1}{20} \int_0^\infty u^3 K_0(u)^6\, \mathrm{d}u - \frac{2^5 - 1}{160}\, \zeta(5) \tag{9}$$

(the $2^5 - 1$ factor multiplying $\zeta(5)$ has to be viewed in parallel with $2^3 - 1$ multiplying $\zeta(3)$ in (3) and $2^2 - 1$ multiplying $\zeta(2)$ in (6)). Note that from now on an identity obtained from a numerical PSLQ search will be labelled $=_{\mathrm{PSLQ}}$ as in (9).

The fact that the double nested integrals in (2) can be reexpressed as linear combination with rational coefficients of simple Bessel integrals with the right weight -here weight 6- and of $\zeta(5)$, fits well in the "Bessel integral $\to$ zeta number" mapping. If a product f of Bessel functions is, like in (3, 4, 5, 6, 7), mapped by simple integration on a zeta number denoted by $\tilde{\zeta}(f)$

$$f \to \int_0^\infty f(u)\mathrm{d}u = \tilde{\zeta}(f)$$

then for a pair of such products f, g the mapping by double "nested" integration

$$f, g \to \int_0^\infty f(u)\mathrm{d}u \int_0^u g(x)\mathrm{d}x = \tilde{\zeta}(f, g)$$

on a polyzeta number denoted by $\tilde{\zeta}(f, g)$ makes sense, since, because of

$$\int_0^\infty f(u)\mathrm{d}u \int_0^u g(x)\mathrm{d}x = \int_0^\infty f(u)\mathrm{d}u \int_0^\infty g(x)\mathrm{d}x - \int_0^\infty g(u)\mathrm{d}u \int_0^u f(x)\mathrm{d}x,$$

one has

$$\tilde{\zeta}(f, g) = \tilde{\zeta}(f)\tilde{\zeta}(g) - \tilde{\zeta}(g, f) \tag{10}$$

in analogy with

$$\zeta(p,q) = \zeta(p)\zeta(q) - \zeta(p+q) - \zeta(q,p) \tag{11}$$

for the standard polyzeta $\zeta(p,q) = \sum_{n>m} \frac{1}{n^p}\frac{1}{m^q}$ - if $\zeta(p,q)$ would be defined as $\sum_{n>m} \frac{1}{n^p}\frac{1}{m^q} + \frac{1}{2}\zeta(p+q)$ then (11) would take the form (10).

2.2 Recurrence

It appears that 1 and $\int_0^\infty u\,K_0(u)^4\,\mathrm{d}u$ one the one hand, and 1, $\int_0^\infty u\,K_0(u)^6\,\mathrm{d}u$ and $\int_0^\infty u^3\,K_0(u)^6\,\mathrm{d}u$ on the other hand, play the role of building blocks in the linear combinations reexpressing, for example, (4) and (8). This might indicate their special role as basis for more general families of Bessel integrals[4].

Indeed, consider such a family of Bessel integrals with a given weight κ

$$I_{n,j}^{(\kappa)} = \frac{1}{n!}\int_0^\infty u^{n+1}K_0(u)^{\kappa-j}K_1(u)^j\mathrm{d}u \qquad j=0,1,\ldots,\kappa \tag{12}$$

where $n \geq \kappa - 1$ for $I_{n,\kappa}^{(\kappa)}$ to be finite. Integration by parts gives the mapping $I_{n,\kappa}^{(\kappa)} \to I_{n+1,\kappa}^{(\kappa)}$

$$I_{n,j}^{(\kappa)} = \frac{n+1}{n-j+2}\left[jI_{n+1,j-1}^{(\kappa)} + (\kappa-j)I_{n+1,j+1}^{(\kappa)}\right] \tag{13}$$

with the $(\kappa+1)\times(\kappa+1)$ matrix

$$\begin{pmatrix}
0 & \frac{\kappa(n+1)}{n+2} & 0 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & \kappa-1 & 0 & 0 & \ldots & 0 & 0 \\
0 & \frac{2(n+1)}{n} & 0 & \frac{(\kappa-2)(n+1)}{n} & 0 & \ldots & 0 & 0 \\
0 & 0 & \frac{3(n+1)}{n-1} & 0 & \frac{(\kappa-3)(n+1)}{n-1} & \ldots & 0 & 0 \\
0 & 0 & 0 & \frac{4(n+1)}{n-2} & 0 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & \frac{n+1}{n-\kappa+3} \\
0 & 0 & 0 & 0 & 0 & \ldots & \frac{\kappa(n+1)}{n-\kappa+2} & 0
\end{pmatrix} \tag{14}$$

whose determinant when κ is odd is $-\frac{(3)^2(5)^2\ldots(\kappa)^2(n+1)^{2\kappa+1}}{(n+2)(n+1)(n)\ldots(n-\kappa+2)}$, and when κ is even is vanishing since in this case

$$\sum_{l=0}^{\kappa/2}(-1)^l(n-2l+2)\binom{\kappa/2}{l}I_{n,2l}^{(\kappa)} = 0 \tag{15}$$

[4]The same is true of 1 and $\int_0^\infty \mathrm{d}u\,uI_0(u)K_0(u)^3$ for (6) and more generally for the family of Bessel integrals obtained from (12) by replacing either a K_0 by a I_0 or a K_1 by a I_1, see [4].

Applying (13) twice gives the mapping $n \to n+2$

$$
\begin{aligned}
I_{n,j}^{(\kappa)} \quad = \quad & \frac{(n+1)(n+2)(j-1)j}{(n-j+2)(n-j+4)} I_{n+2,j-2}^{(\kappa)} \\
& + \frac{(n+1)(n+2)}{n-j+2} \left[\frac{(j+1)(\kappa-j)}{n-j+2} + \frac{j(\kappa-j+1)}{n-j+4} \right] I_{n+2,j}^{(\kappa)} \\
& + \frac{(n+1)(n+2)(\kappa-j-1)(\kappa-j)}{(n-j+2)^2} I_{n+2,j+2}^{(\kappa)}
\end{aligned}
\tag{16}
$$

which conserves the parity of $n-j$. It follows that the $I_{n,j}^{(\kappa)}$'s are divided into two sub-families, depending on the parity of $n-j$.

Let us focus on the sub-family $n-j$ even[5]: it is enough to assume that n is even, and consider

$$
I_{n,0}^{(\kappa)}, \quad I_{n,2}^{(\kappa)}, \quad \ldots, \quad I_{n,\kappa}^{(\kappa)}, \quad n \geq \kappa \quad (\kappa \text{ even}) \tag{17}
$$

or

$$
I_{n,0}^{(\kappa)}, \quad I_{n,2}^{(\kappa)}, \quad \ldots, \quad I_{n,\kappa-1}^{(\kappa)}, \quad n \geq \kappa-1 \quad (\kappa \text{ odd}) \tag{18}
$$

and note that

$$
I_{n+1,1}^{(\kappa)}, \quad I_{n+1,3}^{(\kappa)}, \quad \ldots, \quad I_{n+1,\kappa-1}^{(\kappa)} \quad (\kappa \text{ even}) \tag{19}
$$

or

$$
I_{n+1,1}^{(\kappa)}, \quad I_{n+1,3}^{(\kappa)}, \quad \ldots, \quad I_{n+1,\kappa}^{(\kappa)} \quad (\kappa \text{ odd}) \tag{20}
$$

are respectively related to (17) and (18) by inverting (13), keeping in mind (15) when κ is even[6].

By inverting (16) (again keeping in mind (15)), all integrals in (17) (and thus in (19)) are linear combinations with rational coefficients of the initial conditions

$$
\{I_{\kappa,0}^{(\kappa)}, \quad I_{\kappa,2}^{(\kappa)}, \quad \ldots, \quad I_{\kappa,\kappa}^{(\kappa)}\} \quad (\kappa \text{ even}) \tag{22}
$$

Now, on the one hand from (13) one has $2I_{\kappa-1,\kappa-1}^{(\kappa)} = \kappa(I_{\kappa,\kappa}^{(\kappa)} + (\kappa-1)I_{\kappa,\kappa-2}^{(\kappa)})$ and, on the other hand, $I_{\kappa-1,\kappa-1}^{(\kappa)} = 1/\kappa!$. It follows that $I_{\kappa,\kappa}^{(\kappa)}$ can be replaced by 1. Using (15), one can drop one more element: therefore, all integrals in (17) (and in (19)) are linear combinations with rational coefficients of the $\kappa/2$ numbers

[5] An analysis for the sub-family $n-j$ odd can be done along the same lines.

[6] The integrals (17) and (19) (respectively (18) and (20)) are tantamount to the set

$$
\int_0^\infty u^{n+1} K_0(u)^{\kappa-j} (uK_1(u))^j \, \mathrm{d}u \qquad n \text{ even } \geq 0 \tag{21}
$$

with κ even (respectively κ odd).

$$\{1\,,\quad I^{(\kappa)}_{\kappa,0}\,,\quad I^{(\kappa)}_{\kappa,2}\,,\quad \ldots,\quad I^{(\kappa)}_{\kappa,\kappa-4}\}\quad (\kappa \text{ even}) \tag{23}$$

Likewise, all integrals in (18) (and thus in (20)) are linear combinations with rational coefficients of

$$\{I^{(\kappa)}_{\kappa-1,0},\quad I^{(\kappa)}_{\kappa-1,2},\quad \ldots,\quad I^{(\kappa)}_{\kappa-1,\kappa-1}\}\quad (\kappa \text{ odd}) \tag{24}$$

that is to say, since $I^{(\kappa)}_{\kappa-1,\kappa-1} = 1/\kappa!$, of the $(\kappa+1)/2$ numbers

$$\{1\,,\quad I^{(\kappa)}_{\kappa-1,0}\,,\quad I^{(\kappa)}_{\kappa-1,2}\,,\quad \ldots,\quad I^{(\kappa)}_{\kappa-1,\kappa-3}\}\quad (\kappa \text{ odd}) \tag{25}$$

Finally, by applying (13) appropriately for $0 \le n \le \kappa$, (23) can be mapped on

$$\{1\,,\quad I^{(\kappa)}_{0,0}\,,\quad I^{(\kappa)}_{2,0}\,,\quad \ldots,\quad I^{(\kappa)}_{\kappa-4,0}\}\quad (\kappa \text{ even}) \tag{26}$$

and (25) on

$$\{1\,,\quad I^{(\kappa)}_{0,0}\,,\quad I^{(\kappa)}_{2,0}\,,\quad \ldots,\quad I^{(\kappa)}_{\kappa-3,0}\}\quad (\kappa \text{ odd}) \tag{27}$$

It follows that for κ even the $\kappa/2$ numbers in (26), namely

$$\{1\,,\quad \int_0^\infty u K_0(u)^\kappa \mathrm{d}u\,,\quad \int_0^\infty u^3 K_0(u)^\kappa\,,\quad \ldots,\quad \int_0^\infty u^{\kappa-3} K_0(u)^\kappa\} \tag{28}$$

constitute a basis for the integrals (17) and (19). Likewise for κ odd, the $(\kappa+1)/2$ numbers in (27), namely

$$\{1\,,\int_0^\infty u K_0(u)^\kappa \mathrm{d}u\,,\int_0^\infty u^3 K_0(u)^\kappa\,,\ldots,\int_0^\infty u^{\kappa-2} K_0(u)^\kappa\} \tag{29}$$

constitute a basis for the integrals (18) and (20).

By basis one means that these numbers should be independent over Q: none of them is a linear combination with rational coefficients of the others i.e. there are no positive or negative integers a, a_n such that

$$a \times 1 + \sum_{n \text{ even}=0}^{\kappa-4} a_n \times \int_0^\infty u^{n+1} K_0(u)^\kappa = 0 \quad (\kappa \text{ even}) \tag{30}$$

or such that

$$a \times 1 + \sum_{n \text{ even}=0}^{\kappa-3} a_n \times \int_0^\infty u^{n+1} K_0(u)^\kappa = 0 \quad (\kappa \text{ odd}) \tag{31}$$

This implies that the numbers in (28) with κ even are irrational and also irrational the one relatively to the other. The same should be true of the numbers in (29) with κ odd.

2.3 Asymptotic eigenvalues

In the asymptotics limit $n \to \infty$, the mapping (16) becomes

$$I^{(\kappa)}_{n,j} = (j-1)jI^{(\kappa)}_{n+2,j-2} + (2j(\kappa-j)+\kappa)I^{(\kappa)}_{n+2,j} + (\kappa-j-1)(\kappa-j)I^{(\kappa)}_{n+2,j+2} \quad (32)$$

The $[(\kappa - \kappa \bmod 2)/2+1]$ eigenvalues of the resulting matrix are κ^2, $(\kappa-2)^2$, $(\kappa-4)^2, \ldots$ where the last eigenvalue is 1 for odd κ and 0 for even κ. In the latter case one has to reduce the dimension of the matrix by one unit, using (15). Thereupon, the largest eigenvalue of the inverse matrix is 1 or 1/4 for κ odd or even, respectively, whereas the smallest one is $1/\kappa^2$. Remarkably, with the initial conditions (22), (24), it is the smallest eigenvalue $1/\kappa^2$ that determines the asymptotic behavior of the inverse matrix[7]. This can be understood by noting that the eigenvector corresponding to the eigenvalue κ^2 is $\{1,1,\ldots,1\}$, since $(j-1)\times 1+(2j(\kappa-j)+\kappa)\times 1+(\kappa-j-1)(\kappa-j)\times 1 = \kappa^2$ for all j's. Now, the $I^{(\kappa)}_{n\to\infty,j}$'s happen to not depend on j, because their integrand $u^{n+1}K_0(u)^{\kappa-j}K_1(u)^j$ peaks when $n \to \infty$ at large values of u, where both $K_0(u)$ and $K_1(u)$ are approximated by $K_\nu(u) \underset{u\to\infty}{\longrightarrow} \sqrt{\frac{\pi}{2u}}e^{-u}$. Therefore, in the asymptotic limit, the vector $\{I^{(\kappa)}_{n\to\infty,j}\}$ is indeed proportional to $\{1,1,\ldots,1\}$.

2.4 Question

The irrationality claims on (28) and (29) have been checked numerically. They are also supported by the eigenvalues discussion above: (22) and (24) uniquely lead to an asymptotic behavior governed by the corresponding smallest eigenvalue and are decomposed on the basis (28) and (29). It might mean that the building blocks of (28) and (29), Bessel integrals $\int_0^\infty u^{n+1}\, K_0(u)^\kappa\, \mathrm{d}u$ with n even, play some special role in number theory. Clearly the cases $\kappa=1$ and $\kappa=2$ are trivial since $\int_0^\infty u^{n+1}\, K_0(u)^{1-j}(uK_1(u))^j\, \mathrm{d}u$ with $j=0,1$ and $\int_0^\infty u^{n+1}\, K_0(u)^{2-j}(uK_1(u))^j\, \mathrm{d}u$ with $j=0,1,2$ are all rational and accordingly both the basis (27) for $\kappa=1$ and (26) for $\kappa=2$ have for sole element 1. So the question asked when $\kappa \geq 3$: is it possible to assess the irrationality of the basis (28) and (29)?

One will show in the next section that in the first non trivial cases $\kappa=4$ and $\kappa=3$ where the basis (26) and (27) have dimension 2 the algebra above leads to continous fractions with seemingly insufficient fast numerical convergence to hint at the irrationality of $I^{(4)}_{0,0}$ or $I^{(3)}_{0,0}$.

As far as the case $\kappa=4$ is concerned, this question is formal since $I^{(4)}_{0,0}$ can be integrated to a rational number times $\zeta(3)$, which is known to be

[7]Starting from any other initial condition would lead to an asymptotics governed by the highest eigenvalue.

irrational [7]. When $\kappa = 3$ on the other hand, $I^{(3)}_{0,0}$ can be also be integrated to a rational number times $\psi_1(1/3) - \psi_1(2/3)$, a $\zeta(2)$-like number, whose irrationality is so far not known. Some intimate relation with Apéry's proofs of the irrationality of $\zeta(3)$ and $\zeta(2)$ will show up in the process.

3 CONTINUOUS FRACTIONS

3.1 Weight $\kappa = 4 \to \zeta(3)$

The $n \to n+2$ mapping (16)

$$\begin{pmatrix} I^{(4)}_{n,0} \\ I^{(4)}_{n,2} \\ I^{(4)}_{n,4} \end{pmatrix} = 2(1+n) \begin{pmatrix} \frac{2}{2+n} & \frac{6}{2+n} & 0 \\ \frac{1}{n} & \frac{6(1+n)}{n^2} & \frac{(2+n)}{n^2} \\ 0 & \frac{6(2+n)}{(-2+n)n} & \frac{2(2+n)}{(-2+n)n} \end{pmatrix} \begin{pmatrix} I^{(4)}_{n+2,0} \\ I^{(4)}_{n+2,2} \\ I^{(4)}_{n+2,4} \end{pmatrix}$$

leads to, using $(n+2)I^{(4)}_{n,0} - 2nI^{(4)}_{n,2} + (n-2)I^{(4)}_{n,4} = 0$ in (15)

$$\begin{pmatrix} I^{(4)}_{n,0} \\ I^{(4)}_{n,4} \end{pmatrix} = (1+n) \begin{pmatrix} \frac{32+10n}{(2+n)^2} & \frac{6n}{(2+n)^2} \\ \frac{6(4+n)}{(-2+n)n} & \frac{8+10n}{(-2+n)n} \end{pmatrix} \begin{pmatrix} I^{(4)}_{n+2,0} \\ I^{(4)}_{n+2,4} \end{pmatrix} \tag{33}$$

with asymptotics when $n \to \infty$

$$\begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_+ + \lambda_-}{2} & \frac{\lambda_+ - \lambda_-}{2} \\ \frac{\lambda_+ - \lambda_-}{2} & \frac{\lambda_+ + \lambda_-}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

and eigenvalues $\{\lambda_- = 4, \lambda_+ = 16\}$ and eigenvectors $\{\{1,-1\},\{1,1\}\}$. As already stated in the Introduction the asymptotic scaling is governed by the asymptotic eigenvector $\{1,1\}$ corresponding to the eigenvalue $\lambda_+ = 16$ (and thus to $\frac{1}{16}$ for the inverse iteration). Since n is even set $n = 2k$ and define

$$u(k) = \frac{I^{(4)}_{2k,4}}{I^{(4)}_{2k,0}} \tag{34}$$

to obtain

$$u(k) = \frac{a(k) + c(k)u(k+1)}{b(k) + d(k)u(k+1)} = \frac{c(k)}{d(k)} + \frac{a(k)d(k) - b(k)c(k)}{d(k)(b(k) + d(k)u(k+1))}$$

with

$$\begin{aligned} a(k) &= 3(2+k)(1+k)^2 \\ b(k) &= (8+5k)(-1+k)k \\ c(k) &= (2+5k)(1+k)^2 \\ d(k) &= 3k(-1+k)k \end{aligned} \tag{35}$$

Call

$$d(k)u(k) - c(k) = z(k)$$

and get the iteration $[z(k) \to z(k-1)]$ with the continuous fraction

$$z(k) = \frac{(a(k)d(k) - b(k)c(k))d(k+1)}{d(k)} \frac{1}{d(k+1)(\frac{b(k)}{d(k)} + \frac{c(k+1)}{d(k+1)}) + z(k+1)}$$

This is finally

$$z(k-1) = \frac{-16k^6}{2 + 9k + 15k^2 + 10k^3 + z(k)} \tag{36}$$

with $k \to \infty$ asymptotics

$$z(k) \simeq \frac{-16k^6}{10k^3 + z(k)} \Rightarrow \lim_{k\to\infty} \frac{z(k)}{k^3} = \{-2, -8\}$$

here $\lim_{k\to\infty} z(k)/k^3 = \lim_{k\to\infty} (d(k)u(k) - c(k))/k^3 = -2$ since one iterates from $u(\infty) = 1$. Calling $\tilde{z}(k) = d(k)(\frac{b(k-1)}{d(k-1)} + \frac{c(k)}{d(k)}) + z(k) = d(k)(u(k) + \frac{b(k-1)}{d(k-1)}))$, the inverse iteration is

$$\tilde{z}(k) = \frac{-16k^6}{2 - 9k + 15k^2 - 10k^3 + \tilde{z}(k-1)}$$

with asymptotics

$$\tilde{z}(k) \simeq \frac{-16k^6}{-10k^3 + \tilde{z}(k)} \Rightarrow \lim_{k\to\infty} \frac{\tilde{z}(k)}{k^3} = \{2, 8\}$$

here $\lim_{k\to\infty} \tilde{z}(k)/k^3 = \lim_{k\to\infty} (d(k)u(k) + b(k))/k^3 = 8$ since one iterates from the Bessel integrals initial condition $\tilde{z}(2)$.

Continuous fractions appear in Apéry's proof of $\zeta(3)$ (and $\zeta(2)$) irrationality:

$$\zeta(3) = \frac{6 + 0()}{5 + 1\left(\frac{-1^6}{P(1) + \frac{-2^6}{P(2) + \frac{-3^6}{P(3) + \ldots}}}\right)} \tag{37}$$

with the Apéry polynomial

$$P(k) = 5 + 27k + 51k^2 + 34k^3 = (2k+1)(17k^2 + 17k + 5)$$

This corresponds to the iteration

$$z(k-1) = \frac{-k^6}{P(k) + z(k)} \tag{38}$$

and to the rational approximation

$$\zeta(3) = \lim_{k\to\infty} \frac{y(k)_{\{y(0)=0,y(1)=6\}}}{y(k)_{\{y(0)=1,y(1)=5\}}} \tag{39}$$

with

$$y(k+1) - P(k)y(k) + k^6 y(k-1) = 0 \tag{40}$$

The initial conditions are i.e. $y(0) = 0,\ y(1) = 6 \to y(k)_{\{y(0)=0,y(1)=6\}}$ and $y(0) = 1,\ y(1) = 5 \to y(k)_{\{y(0)=1,y(1)=5\}}$. $\zeta(3)$ is proven to be irrational thanks to the sufficiently fast convergence with respect to the increasing size of the denominators of the rational approximation (39).

In the weight $\kappa = 4$ case the iteration (36) starts at $k = 2$ ($n = 4$). From (36)

$$z(2) = \frac{-16\times 3^6}{2+9k+15k^2+10k^3|_{k=3} + \frac{-16\times 4^6}{2+9k+15k^2+10k^3|_{k=4}+\dots}} = \lim_{k\to\infty} \frac{y(k)_{\{y(2)=1,y(3)=0\}}}{y(k)_{\{y(2)=0,y(3)=1\}}} \tag{41}$$

with

$$y(k+1) - (2+9k+15k^2+10k^3)y(k) + 16k^6 y(k-1) = 0 \tag{42}$$

Using (13) the $k = 2$ ($n = 4$) initial conditions are reexpressed on the basis (28) for $\kappa = 4$, i.e. on $\{1, \quad I^{(4)}_{0,0}\}$, as

$$I^{(4)}_{4,0} = -\frac{9}{512}\times 1 + \frac{7}{384}\times I^{(4)}_{0,0} \quad \text{and} \quad I^{(4)}_{4,4} = \frac{53}{1536}\times 1 - \frac{3}{128}\times I^{(4)}_{0,0}$$

so

$$z(2) = -96\frac{37-36I^{(4)}_{0,0}}{27-28I^{(4)}_{0,0}} = \lim_{k\to\infty} \frac{y(k)_{\{y(2)=1,y(3)=0\}}}{y(k)_{\{y(2)=0,y(3)=1\}}} \tag{43}$$

It appears numerically that the convergence of the rational approximation (41) does not seem sufficiently fast to prove the irrationality of $z(2)$ that is to say that of $I^{(4)}_{0,0}$. One notes in (36)

- in the denominator $2 + 9k + 15k^2 + 10k^3 = P(k) - 3(2k+1)^3$ where $P(k)$ is the Apéry polynomial.
- in the numerator k^6 as in the Apéry case.

These are not coincidences: $I_{0,0}^{(4)} = \int_0^\infty uK_0(u)^4\,du$ has already been integrated in (3) to be proportionnal to $\zeta(3)$

$$I_{0,0}^{(4)} = \frac{2^3-1}{8}\zeta(3) = \frac{\sum_{p=1}^{\infty}\frac{1}{p^3} - \sum_{p=1}^{\infty}\frac{(-1)^p}{p^3}}{2} \tag{44}$$

so that (43) is in fact a rational approximation to $\zeta(3)$

$$z(2) = -96\frac{74-63\zeta(3)}{54-49\zeta(3)} = \lim_{k\to\infty}\frac{y(k)_{\{y(2)=1,y(3)=0\}}}{y(k)_{\{y(2)=0,y(3)=1\}}}$$

One can push further the analogy by rewriting both Apéry and weight $\kappa=4$ rational approximations to $\zeta(3)$ starting from $z(0)$ i.e. with initial conditions $\{y(0)=1,\ y(1)=0\}$ and $\{y(0)=0,\ y(1)=1\}$.

Apéry: from

$$\zeta(3) = \frac{6+0()}{5+1\left(\frac{-1^6}{P(1)+\frac{-2^6}{P(2)+\frac{-3^6}{P(3)+\dots}}}\right)} = \lim_{k\to\infty}\frac{y(k)_{\{y(0)=0,y(1)=6\}}}{y(k)_{\{y(0)=1,y(1)=5\}}}$$

one deduces

$$z(0) = \frac{6}{\zeta(3)} - 5 = \frac{-1^6}{P(1)+\frac{-2^6}{P(2)+\frac{-3^6}{P(3)+\dots}}} = \lim_{k\to\infty}\frac{y(k)_{\{y(0)=1,y(1)=0\}}}{y(k)_{\{y(0)=0,y(1)=1\}}}$$

with $y(k)$ given in (40).

Weight $\kappa=4$: one starts from $z(2)$ in (43) and with (36) iterates to

$$z(0) = \frac{3}{2I_{0,0}^{(4)}} - 2 \tag{45}$$

where a simplification has occured since one would have expected in general

$$z(0) = \frac{a+b\,I_{0,0}^{(4)}}{c+d\,I_{0,0}^{(4)}}$$

Finally one obtains

$$z(0) = \frac{3}{2I_{0,0}^{(4)}} - 2 = \frac{12}{7\zeta(3)} - 2 = \lim_{k\to\infty}\frac{y(k)_{\{y(0)=1,y(1)=0\}}}{y(k)_{\{y(0)=0,y(1)=1\}}}$$

with $y(k)$ given in (42).

So finally one has

- **Apéry:**

$$z(k-1) = \frac{-k^6}{P(k) + z(k)} = \frac{-k^6}{(2k+1)(17k^2+17k+5) + z(k)}$$

with asymptotics

$$z(k) \simeq \frac{-k^6}{34k^3 + z(k)} \Rightarrow \lim_{k\to\infty} \frac{z(k)}{k^3} = \{-(1+\sqrt{2})^4,\ -\frac{1}{(1+\sqrt{2})^4}\}$$

$$y(k+1) - P(k)y(k) + k^6 y(k-1) = 0 \to_{k\to\infty} y^2 - 34y + 1 = 0 \to \{(1+\sqrt{2})^4,\ \frac{1}{(1+\sqrt{2})^4}\}$$

$$z(0) = \frac{42}{7\zeta(3)} - 5 = \lim_{k\to\infty} \frac{y(k)_{\{y(0)=1,y(1)=0\}}}{y(k)_{\{y(0)=0,y(1)=1\}}}$$

- **Weight**[8] $\kappa = 4$**:**

$$z(k-1) = \frac{-16k^6}{P(k) - 3(2k+1)^3 + z(k)} = \frac{-16k^6}{(2k+1)(5k^2+5k+2) + z(k)} \tag{47}$$

with asymptotics

$$z(k) \simeq \frac{-16k^6}{10k^3 + z(k)} \Rightarrow \lim_{k\to\infty} \frac{z(k)}{k^3} = \{-8,\ -2\}$$

$$y(k+1) - (P(k) - 3(2k+1)^3)y(k) + 16k^6 y(k-1) = 0 \to_{k\to\infty} y^2 - 10y + 16 = 0 \to \{8,\ 2\}$$

[8] One could use that $2 + 9k + 15k^2 + 10k^3 = P(k) - 3(2k+1)^3 = (1+2k)(5k^2+5k+2)$ is a multiple of 2 : then

$$z(k-1) = \frac{-4k^6}{\frac{P(k)-3(2k+1)^3}{2} + z(k)} = \frac{-4k^6}{(2k+1)\frac{5k^2+5k+2}{2} + z(k)} \tag{46}$$

$$y(k+1) - \frac{P(k) - 3(2k+1)^3}{2} y(k) + 4k^6 y(k-1) = 0 \to y^2 - 5y + 4 = 0 \to \{4,\ 1\}$$

$$z(0) = \frac{6}{7\zeta(3)} - 1 = \lim_{k\to\infty} \frac{y(k)_{\{y(0)=1,y(1)=0\}}}{y(k)_{\{y(0)=0,y(1)=1\}}}$$

$$z(0)=\frac{12}{7\zeta(3)}-2=\lim_{k\to\infty}\frac{y(k)_{\{y(0)=1,y(1)=0\}}}{y(k)_{\{y(0)=0,y(1)=1\}}}$$

A PSLQ search also gave

$$z(k-1)=\frac{-k^6}{\frac{P(k)-2(2k+1)^3}{3}+z(k)}=\frac{-k^6}{(2k+1)(3k^2+3k+1)+z(k)}$$

with asymptotics

$$z(k)\simeq\frac{-k^6}{6k^3+z(k)}\Rightarrow\lim_{k\to\infty}\frac{z(k)}{k^3}=\{-(1+\sqrt{2})^2,\ -\frac{1}{(1+\sqrt{2})^2}\}$$

$$y(k+1)-\frac{P(k)-2(2k+1)^3}{3}y(k)+k^6y(k-1)=0\to_{k\to\infty}y^2-6y+1=0\to\{(1+\sqrt{2})^2,\ \frac{1}{(1+\sqrt{2})^2}\}$$

$$z(0)=_{\mathrm{PSLQ}}\frac{8}{7\zeta(3)}-1=\lim_{k\to\infty}\frac{y(k)_{\{y(0)=1,y(1)=0\}}}{y(k)_{\{y(0)=0,y(1)=1\}}}$$

(one has used that $P(k)-2(2k+1)^3=(2k+1)(9k^2+9k+3)$ is a multiple of 3).

Only the Apéry rational approximation has a sufficiently fast convergence to check numerically the irrationality of $z(0)=\frac{6}{\zeta(3)}-5$, i.e. of $\zeta(3)$.

3.2 Weight $\kappa=3\to\psi_1(1/3)-\psi_1(2/3)$

The $n\to n+2$ mapping (16) is

$$\begin{pmatrix} I^{(3)}_{n,0}\\ I^{(3)}_{n,2}\end{pmatrix}=(1+n)\begin{pmatrix}\frac{3}{2+n} & \frac{6}{2+n}\\ \frac{2}{n} & \frac{(6+7n)}{n^2}\end{pmatrix}\begin{pmatrix} I^{(3)}_{n+2,0}\\ I^{(3)}_{n+2,2}\end{pmatrix}$$

with asymptotics matrix $n\to\infty$

$$\begin{pmatrix}3&6\\2&7\end{pmatrix}=\begin{pmatrix}\frac{\lambda_++3\lambda_-}{4} & \frac{3(\lambda_+-\lambda_-)}{4}\\ \frac{\lambda_+-\lambda_-}{4} & \frac{3\lambda_++\lambda_-}{4}\end{pmatrix}=\begin{pmatrix}1&1\\1&-\frac{1}{3}\end{pmatrix}\begin{pmatrix}\lambda_+&0\\0&\lambda_-\end{pmatrix}\begin{pmatrix}1&1\\1&-\frac{1}{3}\end{pmatrix}^{-1}$$

with eigenvalues $\{\lambda_-=1,\lambda_+=9\}$ and eigenvectors $\{\{1,-1/3\},\{1,1\}\}$. As already stated in the Introduction the asymptotic scaling is governed by the asymptotic eigenvector $\{1,1\}$ corresponding to the eigenvalue $\lambda_+=9$ (and thus to $\frac{1}{9}$ for the inverse iteration).

Define

$$u(k) = \frac{I^{(3)}_{2k,2}}{I^{(3)}_{2k,0}}$$

call

$$z(k) = d(k)u(k) - c(k)$$

and obtain, using

$$a(k) = 2k(1+k)$$

$$b(k) = 3k^2$$

$$c(k) = (1+k)(3+7k)$$

$$d(k) = 6k^2$$

the iteration $[z(k) \to z(k-1)]$ with the continuous fraction

$$z(k-1) = \frac{-9k^4}{10k^2 + 10k + 3 + z(k)} \tag{48}$$

with asymptotics

$$z(k) \simeq \frac{-9k^4}{10k^2 + z(k)} \Rightarrow \lim_{k\to\infty} \frac{z(k)}{k^2} = \{-1, -9\}$$

here $-k^2$ since one iterates from $u(\infty) = 1$. Calling $\tilde{z}(k) = 2 + 9k + 15k^2 + 10k^3 + z(k)$, the inverse iteration $[\tilde{z}(k-1) \to \tilde{z}(k)]$ is

$$\tilde{z}(k) = \frac{-9k^4}{-3 + 10k - 10k^2 + \tilde{z}(k-1)}$$

with asymptotics

$$\tilde{z}(k) \simeq \frac{-9k^4}{-10k^2 + \tilde{z}(k)} \Rightarrow \lim_{k\to\infty} \frac{\tilde{z}(k)}{k^2} = \{1, 9\}$$

here $9k^2$ since one iterates from the Bessel integrals initial condition $\tilde{z}(1)$.

Using (13) the $k = 1$ ($n = 2$) initial conditions are reexpressed on the basis (29) for $\kappa = 3$, i.e. on $\{1, \quad I^{(3)}_{0,0}\}$, as

$$I^{(3)}_{2,0} = -\frac{1}{3} \times 1 + \frac{2}{3} \times I^{(3)}_{0,0} \qquad I^{(3)}_{2,2} = \frac{1}{6} \times 1 \Rightarrow u(1) \equiv \frac{I^{(3)}_{2,2}}{I^{(3)}_{2,0}} = \frac{1}{-2 + 4I^{(3)}_{0,0}} \tag{49}$$

$$\Rightarrow z(1) = -\frac{-23 + 40I^{(3)}_{0,0}}{-1 + 2I^{(3)}_{0,0}} \Rightarrow z(0) = -3 + \frac{3}{2I^{(3)}_{0,0}}$$

where again a simplification has occured as in (45).

So the rational approximation

$$z(0) = -3 + \frac{3}{2I_{0,0}^{(3)}} = \lim_{k\to\infty} \frac{y(k)_{\{y(0)=1,y(1)=0\}}}{y(k)_{\{y(0)=0,y(1)=1\}}} \tag{50}$$

with

$$y(k+1) - (10k^2+10k+3)y(k) + 9k^4 y(k-1) = 0 \rightarrow y^2 - 10y + 9 = 0 \rightarrow \{9,\ 1\} \tag{51}$$

The polynomial $10k^2+10k+3 = 11k^2+11k+3-k(k+1)$ in the denominator of (48) as well as k^4 in the numerator (with the opposite sign) are again "Apéry like". Indeed Apéry's proof of $\zeta(2)$ irrationality[9] reads

$$z(k-1) = \frac{+k^4}{11k^2 + 11k + 3 + z(k)}$$

$$z(0) = \frac{5}{\zeta(2)} - 3$$

This is not a coincidence: for weight $\kappa = 3$ it is possible to integrate $I_{0,0}^{(3)} = \int_0^\infty u K_0(u)^3 \,\mathrm{d}u$ to be proportionnal to a $\zeta(2)$-like number[10]

$$\int_0^\infty u K_0(u)^3 \,\mathrm{d}u = \frac{\psi_1(1/3) - \psi_1(2/3)}{12}$$

keeping in mind that

$$\zeta(2) = \frac{\psi_1(1/3) + \psi_1(2/3)}{8}$$

where $\psi_n(z)$ is the PolyGamma function

$$\psi_n(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}} \tag{52}$$

Again the numerical convergence of (50) does not seem sufficiently fast to conclude to the irrationality of $z(0)$, that is of $I_{0,0}^{(3)}$, that is of $\psi_1(1/3) - \psi_1(2/3)$, whose irrationality is unknown.

[9] There are two other cases (see van der Poorten's review [7])

$z(k-1) = \frac{+8k^4}{(7k^2+7k+2)+z(k)} \Rightarrow z(0) = \frac{4}{\zeta(2)} - 2$

and

$z(k-1) = \frac{+k^4(4k+1)(4k-1)}{(2k+1)(3k^2+3k+\mathbf{1})+z(k)} \Rightarrow z(0) = \frac{5}{2\zeta(2)} - \mathbf{1}$

We found another case of the last type by a PSLQ search, see Section (3.3) below.

[10] See Appendix A.

3.3 Continuous fractions summary

- $\zeta(2)$**: Apéry, two other cases in Van der Poorten, PSLQ numerical**

  $$z(k-1) = \frac{+k^4}{(11k^2+11k+\mathbf{3})+z(k)} \Rightarrow z(0) = \tfrac{5}{\zeta(2)} - \mathbf{3}$$

  $$z(k-1) = \frac{+8k^4}{(7k^2+7k+\mathbf{2})+z(k)} \Rightarrow z(0) = \tfrac{4}{\zeta(2)} - \mathbf{2}$$

  $$z(k-1) = \frac{+k^4(4k+1)(4k-1)}{(2k+1)(3k^2+3k+\mathbf{1})+z(k)} \Rightarrow z(0) = \tfrac{5}{2\zeta(2)} - \mathbf{1}$$

  $$z(k-1) = \frac{+3k^4(3k+1)(3k-1)}{(2k+1)(13k^2+13k+\mathbf{4})+z(k)} \Rightarrow z(0) =_{\mathrm{PSLQ}} \tfrac{7}{\zeta(2)} - \mathbf{4}$$

- $\psi_1(1/3) - \psi_1(2/3) : \kappa = 3$

  $$z(k-1) = \frac{-9k^4}{(10k^2+10k+\mathbf{3})+z(k)} \Rightarrow z(0) = \tfrac{18}{\psi_1(1/3)-\psi_1(2/3)} - \mathbf{3}$$

- $\zeta(3)$**: Apéry , PSLQ numerical,** $\kappa = 4$

  $$z(k-1) = \frac{-k^6}{(2k+1)(17k^2+17k+\mathbf{5})+z(k)} \Rightarrow z(0) = \tfrac{6}{\zeta(3)} - \mathbf{5}$$

  $$z(k-1) = \frac{-k^6}{(2k+1)(3k^2+3k+\mathbf{1})+z(k)} \Rightarrow z(0) =_{\mathrm{PSLQ}} \tfrac{8}{7\zeta(3)} - \mathbf{1}$$

  $$z(k-1) = \frac{-4k^6}{(2k+1)(\dfrac{5k^2+5k}{2}+\mathbf{1})+z(k)} \Rightarrow z(0) = \tfrac{6}{7\zeta(3)} - \mathbf{1}$$

where one notes

- the usual $pk^2 + pk + \mathbf{q}$ in the denominators and $\mathbf{q}$ in the $z(0)$'s
- the sign change[11] $+k^4 \to -k^4$ in relation with $\psi_1(1/3) + \psi_1(2/3) \to \psi_1(1/3) - \psi_1(2/3)$.

Various PSLQ searches did not produce so far any other example of continuous fractions of this type for $\zeta(2)$, $\zeta(3)$ or $\psi_1(1/3) - \psi_1(2/3)$ (for which only the case[12] listed above seems to be known).

[11] See also (44): from this point of view Apéry's $\frac{6}{\zeta(3)} - 5$ could rather rewrite as $\frac{42}{7\zeta(3)} - 5$.

[12] This continuous fraction appears as a "sporadic" case in the numerical search [9] where it is related to the Dirichlet L-series $L_3(2)$ which "is, up to a factor $3^{3/2}/4$, the maximum volume of a tetrahedron in hyperbolic 3-space".

4 TRYING TO INTEGRATE $\int_0^\infty \mathrm{d}u\, uK_0(u)^n$

As alluded to in Section (2) Feyman diagrams momenta integrations lead to multiple integrals on differences of consecutive intermediate temperatures. Bessel integrals $\int_0^\infty \mathrm{d}u\, uK_0(u)^n$ can quite generally be represented in this particular way: change variable $u = 2\sqrt{t}$ and use the integral representation

$$\int_0^\infty \mathrm{d}a\, a^{\nu-1} e^{-a-\frac{t}{a}} = 2K_\nu(u)(\frac{u}{2})^\nu \tag{53}$$

so that

$$\begin{aligned}
&\int_0^\infty \mathrm{d}u\, uK_0(u)^n \\
=&\int_0^\infty 2\mathrm{d}t \int_0^\infty \frac{\mathrm{d}a_1}{2} \ldots \int_0^\infty \frac{\mathrm{d}a_n}{2} \frac{1}{a_1 a_2 \ldots a_n} e^{-a_1-\frac{t}{a_1}} e^{-a_2-\frac{t}{a_2}} \ldots e^{-a_n-\frac{t}{a_n}} \\
=&2\int_0^\infty \frac{\mathrm{d}a_1}{2} \ldots \int_0^\infty \frac{\mathrm{d}a_n}{2} \frac{1}{a_1 a_2 \ldots a_n} \frac{1}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}} e^{-a_1-a_2-\ldots-a_n} \\
=&\frac{1}{2^{n-1}} \int_0^\infty \mathrm{d}a_1 \ldots \int_0^\infty \mathrm{d}a_n \frac{1}{a_1 a_2 \ldots a_{n-1} + a_2 a_3 \ldots a_n + \ldots + a_n a_1 \ldots a_{n-2}} e^{-(a_1+a_2+\ldots+a_n)}
\end{aligned} \tag{54}$$

a_n **integration:** introduce the variable β to rewrite $e^{-(a_1+a_2+\ldots+a_n)} = \int_0^\infty \mathrm{d}\beta\, e^{-\beta} \delta(\beta - (a_1+a_2+\ldots+a_n))$, then integrate over a_n

$$\begin{aligned}
&\int_0^\infty \mathrm{d}u\, uK_0(u)^n \\
=&\frac{1}{2^{n-1}} \int_0^\infty \mathrm{d}\beta e^{-\beta} \int_0^\beta \mathrm{d}a_1 \int_0^{\beta-a_1} \mathrm{d}a_2 \ldots \int_0^{\beta-a_1-\ldots-a_{n-2}} \mathrm{d}a_{n-1} \\
&\frac{1}{a_1 a_2 \ldots a_{n-1} + (\beta-(a_1+a_2+\ldots+a_{n-1}))(a_2 a_3 \ldots a_{n-1} + a_3 a_4 \ldots a_{n-1} a_1 + \ldots + a_1 \ldots a_{n-2})}
\end{aligned} \tag{55}$$

Change variables $a_i \to a'_i = a_i/\beta$ (notation $a'_i \to a_i$) so that the β integration becomes trivial and finally

$$\begin{aligned}
&\int_0^\infty \mathrm{d}u\, uK_0(u)^n = \frac{1}{2^{n-1}} \int_0^1 \mathrm{d}a_1 \int_0^{1-a_1} \mathrm{d}a_2 \ldots \int_0^{1-a_1-\ldots-a_{n-2}} \mathrm{d}a_{n-1} \\
&\frac{1}{a_1 a_2 \ldots a_{n-1} + (1-(a_1+a_2+\ldots+a_{n-1}))(a_2 a_3 \ldots a_{n-1} + a_3 a_4 \ldots a_{n-1} a_1 + \ldots + a_1 a_2 \ldots a_{n-2})}
\end{aligned} \tag{56}$$

With the notations

$u_n = a_1 + a_2 + \ldots + a_n$

$v_n = a_2 a_3 \ldots a_n + a_3 a_4 \ldots a_n a_1 + \ldots + a_n a_1 a_2 \ldots a_{n-2} + a_1 a_2 \ldots a_{n-1}$

$w_n = a_1 a_2 \ldots a_n$

one has shown that

$$\int_0^\infty \mathrm{d}u\, uK_0(u)^n = \frac{1}{2^{n-1}}\int_0^\infty \mathrm{d}a_1 \int_0^\infty \mathrm{d}a_2 \ldots \int_0^\infty \mathrm{d}a_n \frac{1}{v_n} e^{-u_n}$$
$$= \frac{1}{2^{n-1}}\int_0^1 \mathrm{d}a_1 \int_0^{1-u_1} \mathrm{d}a_2 \ldots \int_0^{1-u_{n-2}} \mathrm{d}a_{n-1} \frac{1}{w_{n-1}+(1-u_{n-1})v_{n-1}} \tag{57}$$

a_{n-1} **integration:** use

$$u_{n-1} = a_{n-1} + u_{n-2}, \quad v_{n-1} = a_{n-1}v_{n-2} + w_{n-2}, \quad w_{n-1} = a_{n-1}w_{n-2}$$

so that

$$\frac{1}{w_{n-1}+(1-u_{n-1})v_{n-1}} = \frac{1}{w_{n-2}(1-u_{n-2}) + a_{n-1}v_{n-2}(1-u_{n-2}) - a_{n-1}^2 v_{n-2}}$$
$$= \frac{1}{-v_{n-2}(a_{n-1}-a_{n-1}^+)(a_{n-1}-a_{n-1}^-)} = \frac{1}{-v_{n-2}(a_{n-1}^+-a_{n-1}^-)}\left(\frac{1}{a_{n-1}-a_{n-1}^+} - \frac{1}{a_{n-1}-a_{n-1}^-}\right) \tag{58}$$

where

$$a_{n-1}^\pm = \frac{-v_{n-2}(1-u_{n-2}) \pm \sqrt{v_{n-2}^2(1-u_{n-2})^2 + 4w_{n-2}v_{n-2}(1-u_{n-2})}}{-2v_{n-2}}$$

One finds

$$\int_0^\infty \mathrm{d}u\, uK_0(u)^n = \frac{1}{2^{n-1}}\int_0^1 \mathrm{d}a_1 \int_0^{1-u_1} \mathrm{d}a_2 \ldots \int_0^{1-u_{n-3}} \mathrm{d}a_{n-2}$$
$$\frac{1}{\sqrt{v_{n-2}^2(1-u_{n-2})^2 + 4w_{n-2}v_{n-2}(1-u_{n-2})}} \log\left|\frac{a_{n-1}^-}{a_{n-1}^+}\frac{1-u_{n-2}-a_{n-1}^+}{1-u_{n-2}-a_{n-1}^-}\right| \tag{59}$$

where the integrand rewrites as

$$\frac{2}{\sqrt{v_{n-2}^2(1-u_{n-2})^2 + 4w_{n-2}v_{n-2}(1-u_{n-2})}} \log\left|\frac{a_{n-1}^-}{a_{n-1}^+}\right| = \frac{2}{(1-u_{n-2})v_{n-2}} X \log\frac{1+X}{1-X} \tag{60}$$

with

$$X = \sqrt{\frac{1}{1+\frac{4w_{n-2}}{(1-u_{n-2})v_{n-2}}}}$$

Change variables $a_i \to x_i = Xa_i/u_{n-2}$ so that when $0 < a_i < 1$ then $1 > X > 0 \Rightarrow 1 > x_i > 0$. The Jacobian is

$$\frac{(u_{n-2})^{n-2}}{X^{n-3}|\sum_{i=1}^{n-2} a_i\partial_i X|}$$

Use homogeneity relations

$$
\begin{aligned}
\sum_{i=1}^{n-2} a_i \partial_i u_{n-2} &= u_{n-2} \\
\sum_{i=1}^{n-2} a_i \partial_i v_{n-2} &= (n-3) v_{n-2} \\
\sum_{i=1}^{n-2} a_i \partial_i w_{n-2} &= (n-2) w_{n-2}
\end{aligned} \tag{61}
$$

so that

$$
\sum_{i=1}^{n-2} a_i \partial_i X = -\frac{1}{2} X (1-X^2) \frac{1}{1-u_{n-2}}
$$

and

$$
\begin{aligned}
\int_0^\infty \mathrm{d}u \, u K_0(u)^n =& \frac{1}{2^{n-1}} \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \ldots \int_0^{1-x_1-\ldots-x_{n-3}} \mathrm{d}x_{n-2} \log \frac{1+X}{1-X} \\
& \frac{2}{(1-u_{n-2})v_{n-2}} X \frac{(u_{n-2})^{n-2}}{X^{n-3} |-\frac{1}{2} X (1-X^2) \frac{1}{1-u_{n-2}}|} \\
=& \frac{1}{2^{n-1}} \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \ldots \int_0^{1-x_1-\ldots-x_{n-3}} \mathrm{d}x_{n-2} \log \frac{1+X}{1-X} \frac{4}{v_{n-2}} \frac{(u_{n-2})^{n-2}}{X^{n-3}(1-X^2)}
\end{aligned} \tag{62}
$$

Use

$$
\begin{aligned}
w_{n-2} X^{n-2} &= (u_{n-2})^{n-2} x_1 x_2 \ldots x_{n-2} \\
\frac{1}{X^2} - 1 &= \frac{4 w_{n-2}}{(1-u_{n-2}) v_{n-2}}
\end{aligned} \tag{63}
$$

so that

$$
\int_0^\infty \mathrm{d}u \, u K_0(u)^n = \frac{1}{2^{n-1}} \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \ldots \int_0^{1-x_1-\ldots-x_{n-3}} \mathrm{d}x_{n-2} \log \frac{1+X}{1-X} \frac{1}{X} \frac{1-u_{n-2}}{x_1 x_2 \ldots x_{n-2}}
$$

Use

$$
\begin{aligned}
X =& x_1 + x_2 + \ldots + x_{n-2} \\
\frac{u_{n-2} v_{n-2}}{w_{n-2} X} =& \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_{n-2}}
\end{aligned}
$$

and (63) rewritten as

$$
\frac{4 w_{n-2}}{(1-u_{n-2}) v_{n-2}} X = \frac{1}{(x_1 + x_2 + \ldots + x_{n-2})^2} - 1
$$

so that finally

$$
\begin{aligned}
\int_0^\infty \mathrm{d}u \, u K_0(u)^n = \frac{1}{2^{n-1}} \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \ldots \int_0^{1-x_1-\ldots-x_{n-3}} \mathrm{d}x_{n-2} \log \frac{1 + x_1 + x_2 + \ldots + x_{n-2}}{1 - (x_1 + x_2 + \ldots + x_{n-2})} \\
\frac{4}{4(x_1 + \ldots + x_{n-2}) x_1 \ldots x_{n-2} + (1 - (x_1 + \ldots + x_{n-2})^2)(x_2 x_3 \ldots x_{n-2} + \ldots + x_1 x_2 \ldots x_{n-3})}
\end{aligned} \tag{64}
$$

This integration generalises to $\int_0^\infty \mathrm{d}u\, u^p K_0(u)^n$ with p odd: for example

$$\int_0^\infty \mathrm{d}u\, u^3 K_0(u)^n = \frac{1}{2^{n-3}} \int_0^1 \mathrm{d}a_1 \int_0^{1-a_1} \mathrm{d}a_2 \ldots \int_0^{1-a_1-\ldots-a_{n-2}} \mathrm{d}a_{n-1}$$
$$\frac{a_1 a_2 \ldots a_{n-1}(1-(a_1+a_2+\ldots+a_{n-1}))}{(a_1 a_2 \ldots a_{n-1} + (1-(a_1+a_2+\ldots+a_{n-1}))(a_2 a_3 \ldots a_{n-1} + a_3 a_4 \ldots a_{n-1} a_1 + \ldots + a_1 a_2 \ldots a_{n-2}))^2}$$
$$= \frac{1}{2^{n-3}} \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \ldots \int_0^{1-x_1-\ldots-x_{n-3}} \mathrm{d}x_{n-2}$$
$$\frac{\left(\frac{1+(x_1+x_2+\ldots+x_{n-2})^2}{2(x_1+x_2+\ldots+x_{n-2})} \log \frac{1+x_1+x_2+\ldots+x_{n-2}}{1-(x_1+x_2+\ldots+x_{n-2})} - 1\right) 4x_1 \ldots x_{n-2}\left(1-(x_1+x_2+\ldots+x_{n-2})^2\right)}{\left(4(x_1+\ldots+x_{n-2})x_1 \ldots x_{n-2} + (1-(x_1+\ldots+x_{n-2})^2)(x_2 x_3 \ldots x_{n-2} + \ldots + x_1 x_2 \ldots x_{n-3})\right)^2} \tag{65}$$

Finally if in (64) and (65) one changes notations $x_i \to a_i$ one has shown

$$2^{n-1} \int_0^\infty \mathrm{d}u\; u K_0(u)^n = \int_0^\infty \mathrm{d}a_1 \ldots \int_0^\infty \mathrm{d}a_{n-2} \int_0^\infty \mathrm{d}a_{n-1} \int_0^\infty \mathrm{d}a_n \quad \frac{1}{v_n} \quad e^{-u_n}$$
$$= \int_0^1 \mathrm{d}a_1 \int_0^{1-u_1} \mathrm{d}a_2 \ldots \int_0^{1-u_{n-3}} \mathrm{d}a_{n-2} \int_0^{1-u_{n-2}} \mathrm{d}a_{n-1} \quad \frac{1}{w_{n-1} + (1-u_{n-1})v_{n-1}} \tag{66}$$
$$= \int_0^1 \mathrm{d}a_1 \int_0^{1-u_1} \mathrm{d}a_2 \ldots \int_0^{1-u_{n-3}} \mathrm{d}a_{n-2} \quad \log \frac{1+u_{n-2}}{1-u_{n-2}} \quad \frac{4}{4u_{n-2}w_{n-2} + (1-u_{n-2}^2)v_{n-2}} \tag{67}$$

and

$$2^{n-3} \int_0^\infty \mathrm{d}u\; u^3 K_0(u)^n = \int_0^\infty \mathrm{d}a_1 \ldots \int_0^\infty \mathrm{d}a_{n-2} \int_0^\infty \mathrm{d}a_{n-1} \int_0^\infty \mathrm{d}a_n \quad \frac{w_n}{v_n^2} \quad e^{-u_n}$$
$$= \int_0^1 \mathrm{d}a_1 \int_0^{1-u_1} \mathrm{d}a_2 \ldots \int_0^{1-u_{n-3}} \mathrm{d}a_{n-2} \int_0^{1-u_{n-2}} \mathrm{d}a_{n-1} \quad \frac{w_{n-1}(1-u_{n-1})}{(w_{n-1} + (1-u_{n-1})v_{n-1})^2} \tag{68}$$
$$= \int_0^1 \mathrm{d}a_1 \int_0^{1-u_1} \mathrm{d}a_2 \ldots \int_0^{1-u_{n-3}} \mathrm{d}a_{n-2} \left(\frac{1+u_{n-2}^2}{2u_{n-2}} \log \frac{1+u_{n-2}}{1-u_{n-2}} - 1\right) \frac{4w_{n-2}(1-u_{n-2}^2)}{(4u_{n-2}w_{n-2} + (1-u_{n-2}^2)v_{n-2})^2}$$

(66) and (68) indicate that $\int_0^\infty \mathrm{d}u\; u K_0(u)^n$ and $\int_0^\infty \mathrm{d}u\; u^3 K_0(u)^n$ are periods. This is the case in general for $\int_0^\infty \mathrm{d}u\; u^m K_0(u)^n$ with m odd.

5 CONCLUSION : WHY LOOKING AT BESSEL INTEGRALS ? IS $\zeta(5)$ IRRATIONAL ?

In Section (2) some arguments (and examples) were given of why weight κ Bessel integrals are expected to be related to weight $\kappa - 1$ (and below) zeta numbers.

One has found the surprisingly simple PSLQ identity

$$\zeta(5) =_{\text{PSLQ}} \frac{1}{77} \int_0^\infty u\, K_0(u)^8\, \mathrm{d}u - \frac{72}{77} \int_0^\infty u^3\, K_0(u)^8\, \mathrm{d}u \tag{69}$$

If one could prove for $\kappa = 8$ the irrationality claim on (28), in particular on $\int_0^\infty u\, K_0(u)^8\, du$ and $\int_0^\infty u^3\, K_0(u)^8\, du$, and derive (69), then one would have a proof of the irrationality of $\zeta(5)$. A related issue is to generalize the algebraic construction of Section (3) to weights $\kappa \geq 5$, with Bessel basis of dimension ≥ 3 (see Appendix B).

Coming back in Section (3.3) to $\zeta(3)$, $\zeta(2)$ and $\psi_1(1/3) - \psi_1(2/3)$, it would be rewarding to find a systematics beyond the cases listed and have simple expressions (Apéry-like numbers) for the y_n's satisfying the recurrence relations associated to these continuous fractions with initial conditions $\{y(0) = 1, y(1) = 0\}$ or $\{y(0) = 0, y(1) = 1\}$. With the Apéry initial conditions $\{y(0) = 1, y(0) = q\}$, the recurrence $y(k+1) - (2k+1)(3k^2+3k+1)y(k) + k^6 y(k-1) = 0$ associated to the continuous fraction

$$z(k-1) = \frac{-k^6}{(2k+1)(3k^2+3k+\mathbf{1}) + z(k)} \Rightarrow z(0) =_{\text{PSLQ}} \frac{8}{7\zeta(3)} - \mathbf{1}$$

has for solution [10]

$$y(k)_{\{y(0)=1,y(1)=1\}} = \frac{k!^3}{2^{2k}} \sum_{i=0}^{k} \binom{k}{i}^2 \binom{2i}{k}^2 \tag{70}$$

where $k!^3$ factorises. Similarly for the $\kappa = 4$ continuous fraction

$$z(k-1) = \frac{-16k^6}{(2k+1)(5k^2+5k+\mathbf{2}) + z(k)} \Rightarrow z(0) = \frac{12}{7\zeta(3)} - \mathbf{2} \tag{71}$$

$$y(k)_{\{y(0)=1,y(1)=2\}} = \frac{k!^3}{2^k} \sum_{i=0}^{k} \binom{k}{i}^2 \binom{2i}{i} \binom{2(k-i)}{k-i} \tag{72}$$

(see in [9] the solutions for the second $\zeta(2)$ and the $\kappa = 3$ continuous fractions listed in Section (3.3)).

One would also like to integrate $\int_0^\infty u\, K_0(u)^n\, du$ further up to possibly one integration left on an integrand which should contain a term of the type $(\log(1+x)/(1-x))^{n-2}$ times a rational function of x yet to be determined, possibly allowing for a derivation of (69).

Finally the fact that Bessel integrals fall in the category of periods might also be an indication of a deeper meaning yet to be understood.

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APPENDIX A: INTEGRATING FURTHER

1) Trying to integrate $\int_0^\infty \mathbf{d}u\, uI_0(u)K_0(u)^n$

Using again the integral representation (53) and ($u = 2\sqrt{t}$)

$$I_0(u) = \sum_{k=0}^{\infty} \frac{t^k}{(k!)^2} \tag{73}$$

one can integrate over t using

$$\int_0^\infty \mathrm{d}t\, t^k e^{-tx} = \frac{k!}{x^{k+1}} \tag{74}$$

to obtain

$$2^{n-1} \int_0^\infty \mathrm{d}u\; uI_0(u)K_0(u)^n = \int_0^\infty \mathrm{d}a_1 \ldots \int_0^\infty \mathrm{d}a_{n-2} \int_0^\infty \mathrm{d}a_{n-1} \int_0^\infty \mathrm{d}a_n \quad \frac{1}{v_n} \quad e^{-u_n + \dfrac{w_n}{v_n}}$$

Introducing as above the variable β, integrating over a_n and then trivially over β one finally obtains

$$2^{n-1} \int_0^\infty \mathrm{d}u\; uI_0(u)K_0(u)^n$$
$$= \int_0^1 \mathrm{d}a_1 \int_0^{1-u_1} \mathrm{d}a_2 \ldots \int_0^{1-u_{n-3}} \mathrm{d}a_{n-2} \int_0^{1-u_{n-2}} \mathrm{d}a_{n-1} \quad \frac{1}{w_{n-1}u_{n-1} + (1-u_{n-1})v_{n-1}} \tag{75}$$

a result to be compared to (57).

One can push the integration one step further following the same procedure as in Section (4) to obtain an expression again in terms of u_{n-2}, v_{n-2} and w_{n-2} but somehow more involved than (67) since its denominator contains a square root.

2) More on integrating $\int_0^\infty \mathbf{d}u\, uK_0(u)^n$, $\int_0^\infty \mathbf{d}u\, u^3K_0(u)^n$ and $\int_0^\infty \mathbf{d}u\, uI_0(u)K_0(u)^n$

n = 3

$$\int_0^\infty \mathrm{d}u\, uK_0(u)^3 = \frac{1}{2^2} \int_0^1 \mathrm{d}x_1 \log \frac{1+x_1}{1-x_1} \frac{4}{4(x_1)x_1 + (1-(x_1)^2)} = \int_0^1 \mathrm{d}x_1 \log \frac{1+x_1}{1-x_1} \frac{1}{1+3x_1^2}$$

n = 4

$$\int_0^\infty \mathrm{d}u\, uK_0(u)^4 = \frac{1}{2^3} \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \log \frac{1+x_1+x_2}{1-x_1-x_2} \frac{1}{x_1+x_2} \frac{4}{4x_1x_2 + (1-(x_1+x_2)^2)} \tag{76}$$

n = 5

$$\int_0^\infty \mathrm{d}u\, uK_0(u)^5 = \frac{1}{2^4}\int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \int_0^{1-x_1-x_2} \mathrm{d}x_3 \log\frac{1+x_1+x_2+x_3}{1-(x_1+x_2+x_3)} \frac{4}{4(x_1+x_2+x_3)x_1x_2x_3 + (1-(x_1+x_2+x_3)^2)(x_2x_3+x_3x_1+x_1x_2)} \tag{77}$$

Integrating further

n = 3

$$\begin{aligned}\int_0^\infty \mathrm{d}u\, uK_0(u)^3 &= \int_0^1 \mathrm{d}x_1 \log\frac{1+x_1}{1-x_1}\frac{1}{1+3x_1^2} = \int_0^1 \mathrm{d}x_1 \log\frac{1+x_1}{1-x_1}\frac{2}{(1+x_1)^3(1+(\frac{1-x_1}{1+x_1})^3)} \\ &= -\frac{1}{2}\int_0^1 \mathrm{d}u \frac{1+u}{1+u^3}\log u \\ &= -\frac{1}{2}\int_0^1 \mathrm{d}u(1+u)\sum_{k=0}^\infty (-1)^k u^{3k}\log u \\ &= \frac{1}{2}\sum_{k=0}^\infty (-1)^k(\frac{1}{(3k+1)^2}+\frac{1}{(3k+2)^2}) = \frac{\psi_1(1/3)-\psi_1(2/3)}{12}\end{aligned} \tag{78}$$

where one has made the change of variable $u = \frac{1-x_1}{1+x_1}$ and used

$$\int_0^1 \mathrm{d}u\, u^k(\log u)^n = (-1)^n\frac{n!}{(k+1)^n} \tag{79}$$

n = 4

$$\begin{aligned}\int_0^\infty \mathrm{d}u\, uK_0(u)^4 &= \frac{1}{2^3}\int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \log\frac{1+x_1+x_2}{1-x_1-x_2}\frac{1}{x_1+x_2}\frac{4}{1-(x_1+x_2)^2+4x_1x_2} \\ &= \frac{1}{2^3}\frac{1}{2}\int_0^1 \mathrm{d}x\frac{1}{x}\log\frac{1+x}{1-x}\int_{-x}^x \mathrm{d}y\frac{4}{1-y^2} \\ &= \frac{1}{2}\int_0^1 \mathrm{d}x\frac{1}{x}\log\frac{1+x}{1-x}\int_0^x \mathrm{d}y\frac{1}{1-y^2} \\ &= \frac{1}{2}\frac{1}{2}\int_0^1 \mathrm{d}x\frac{1}{x}(\log\frac{1+x}{1-x})^2 = \frac{7\zeta(3)}{8}\end{aligned} \tag{80}$$

where one has made the change of variables $x = x_1+x_2$ and $y = x_1-x_2$ and used

$$\frac{2^s-1}{2^{s-1}}\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}+(-1)^{n-1}\frac{1}{n^s} = \frac{1}{2}\int_0^\infty \mathrm{d}x x^{s-1}(\frac{1}{e^x-1}+\frac{1}{e^x+1}) = \frac{1}{(s-1)!}\int_0^1 \frac{\mathrm{d}x}{x}(\log\frac{1+x}{1-x})^{s-1}$$

or rather, from (80),

$$
\begin{aligned}
\int_0^\infty \mathrm{d}u\, u K_0(u)^4 &= \frac{1}{4}\int_0^1 \mathrm{d}x \frac{1}{x}(\log\frac{1+x}{1-x})^2 = \frac{1}{2}\int_0^1 \mathrm{d}u \frac{1}{1-u^2}(\log u)^2 \\
&= \frac{1}{2}\int_0^1 \mathrm{d}u (\log u)^2 \sum_{n=0}^{\infty} u^{2n} = \frac{7\zeta(3)}{8}
\end{aligned}
\tag{81}
$$

with the change of variable $u = \frac{1-x}{1+x}$.

One knows for example that $\int_0^\infty \mathrm{d}u\, u^3 K_0(u)^4$ is a linear combination with rational coefficients of 1 and $\int_0^\infty \mathrm{d}u\, u^3 K_0(u)^4$ namely

$$
4\int_0^\infty \mathrm{d}u\, u K_0(u)^4 - 16\int_0^\infty \mathrm{d}u\, u^3 K_0(u)^4 = 3 \tag{82}
$$

One has

$$
\begin{aligned}
\int_0^\infty \mathrm{d}u\, u^3 K_0(u)^4 = \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \left(\frac{1+(x_1+x_2)^2}{2(x_1+x_2)} \log\frac{1+x_1+x_2}{1-(x_1+x_2)} - 1\right) \frac{1-(x_1+x_2)^2}{(x_1+x_2)^2} \\
\frac{4x_1x_2}{\left(1-(x_1+x_2)^2+4x_1x_2\right)^2}
\end{aligned}
\tag{83}
$$

so that

$$
\begin{aligned}
&\int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \log\frac{1+x_1+x_2}{1-x_1-x_2} \frac{1}{x_1+x_2} \frac{2}{1-(x_1+x_2)^2+4x_1x_2} \\
&- 8\left(\frac{1+(x_1+x_2)^2}{2(x_1+x_2)} \log\frac{1+x_1+x_2}{1-(x_1+x_2)} - 1\right) \frac{1-(x_1+x_2)^2}{(x_1+x_2)^2} \frac{4x_1x_2}{\left(1-(x_1+x_2)^2+4x_1x_2\right)^2} = 3
\end{aligned}
\tag{84}
$$

has to be satisfied. With the change of variables $x = x_1 + x_2$ and $y = x_1 - x_2$

$$
\begin{aligned}
&\int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \left(\frac{1+(x_1+x_2)^2}{2(x_1+x_2)} \log\frac{1+x_1+x_2}{1-(x_1+x_2)} - 1\right) \frac{1-(x_1+x_2)^2}{(x_1+x_2)^2} \frac{4x_1x_2}{\left(1-(x_1+x_2)^2+4x_1x_2\right)^2} \\
&= \frac{2}{2}\int_0^1 \mathrm{d}x \left(\frac{1+x^2}{2x}\log\frac{1+x}{1-x} - 1\right)\frac{1-x^2}{x^2}\int_0^x \mathrm{d}y \frac{(x+y)(x-y)}{(1-y^2)^2} \\
&= \int_0^1 \mathrm{d}x \left(\frac{1+x^2}{2x}\log\frac{1+x}{1-x} - 1\right)\frac{1-x^2}{x^2}\frac{1}{2}\left(-x + (1+x^2)\frac{1}{2}\log\frac{1+x}{1-x}\right) \\
&= \int_0^1 \mathrm{d}x \frac{1-x^2}{2x}\left(\frac{1+x^2}{2x}\log\frac{1+x}{1-x} - 1\right)^2
\end{aligned}
\tag{85}
$$

so that (84) becomes

$$
\int_0^1 \mathrm{d}x \left(\frac{1}{x}(\log\frac{1+x}{1-x})^2 - 4\frac{1-x^2}{x}(\frac{1+x^2}{2x}\log\frac{1+x}{1-x} - 1)^2\right) = 3 \tag{86}
$$

which is indeed true.

Finally

$$\int_0^\infty \mathrm{d}u\, uI_0(u)K_0(u)^4 = \frac{1}{4}\int_0^1 \mathrm{d}a_1 \int_0^{1-a_1} \mathrm{d}a_2 \frac{1}{a_1a_2(a_1+a_2)-(a_1+a_2)(1-a_1-a_2)} \tag{87}$$

With the change of variables $x = a_1 + a_2$ and $y = a_1 - a_2$ one gets

$$\begin{aligned}
\int_0^\infty \mathrm{d}u\, uI_0(u)K_0(u)^4 =& \frac{1}{4}\int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \frac{4}{x}\frac{1}{x-2+y}\frac{1}{x-2-y} \\
=& \frac{1}{2}\int_0^1 \mathrm{d}x \frac{1}{x(x-2)}\int_0^x \mathrm{d}y \frac{1}{x-2+y} + \frac{1}{x-2-y} \\
=& \frac{1}{2}\int_0^1 \mathrm{d}x \frac{1}{x(x-2)}\log(1-x) \\
=& -\frac{1}{2}\int_0^1 \mathrm{d}u \frac{1}{1-u^2}\log u = \frac{3\zeta(2)}{8}
\end{aligned} \tag{88}$$

where $u = 1 - x$.

3) K-Bessel integrals summary

$$\int_0^\infty \mathrm{d}u\, uK_0(u) = 1$$

$$\int_0^\infty \mathrm{d}u\, K_0(u) = \frac{\pi}{2} = -\frac{\sqrt{3}}{2}(\psi_0(1/3) - \psi_0(2/3))$$

$$\int_0^\infty \mathrm{d}u\, uK_0(u)^2 = \frac{1}{2}$$

$$\int_0^\infty \mathrm{d}u\, K_0(u)^2 = \frac{3}{2}\zeta(2)$$

$$\int_0^\infty \mathrm{d}u\, uK_0(u)^3 = \frac{\psi_1(1/3) - \psi_1(2/3)}{12}$$

$$\int_0^\infty \mathrm{d}u\, K_0(u)^3 = ?$$

$$\int_0^\infty \mathrm{d}u\, uK_0(u)^4 = \frac{7}{8}\zeta(3)$$

$$\int_0^\infty \mathrm{d}u\, K_0(u)^4 = ?$$

$$\ldots ?$$

$$\int_0^\infty \mathrm{d}u\, u K_0(u)^8 - 72 \int_0^\infty \mathrm{d}u\, u^3 K_0(u)^8 =_{\mathrm{PSLQ}} 77\zeta(5)$$

$$\ldots ?$$

$$\lim_{n\to\infty} \frac{2^{n-1}}{n!} \int_0^\infty \mathrm{d}u\, u K_0(u)^n = e^{2\psi_0(1)}$$

$$\lim_{n\to\infty} \frac{1}{n!} \int_0^\infty \mathrm{d}u\, K_0(u)^n = 2e^{\psi_0(1)}$$

where $\psi_0(1)$ is minus the Euler constant.

APPENDIX B: HIGHER ORDER RECURSIONS

From the iteration (16) it is easy to get for $\kappa \geq 5$ higher order recursions generalizing the recursions (42, 51) for $\kappa = 4, 3$. For example in the $\kappa = 4$ case, (16) or (33) becomes, defining $x(k) = I^{(4)}_{2k,0}$, $y(k) = I^{(4)}_{2k,4}$,

$$\begin{pmatrix} x(k) \\ y(k) \end{pmatrix} = \frac{1+2k}{(k-1)k(k+1)^2} \begin{pmatrix} b(k) & d(k) \\ a(k) & c(k) \end{pmatrix} \begin{pmatrix} x(k+1) \\ y(k+1) \end{pmatrix} \tag{89}$$

where $a(k), b(k), c(k), d(k)$ are given in (35).

One can for example start from the inverse iteration

$$\begin{pmatrix} x(k+1) \\ y(k+1) \end{pmatrix} = \begin{pmatrix} \alpha(k) & \beta(k) \\ \gamma(k) & \delta(k) \end{pmatrix} \begin{pmatrix} x(k) \\ y(k) \end{pmatrix} \tag{90}$$

where

$$\begin{aligned}
\alpha(k) &= \frac{(k-1)k(k+1)^2}{1+2k} \frac{1}{b(k)c(k) - a(k)d(k)} c(k) \\
\beta(k) &= -\frac{(k-1)k(k+1)^2}{1+2k} \frac{1}{b(k)c(k) - a(k)d(k)} d(k) \\
\gamma(k) &= -\frac{(k-1)k(k+1)^2}{1+2k} \frac{1}{b(k)c(k) - a(k)d(k)} a(k) \\
\delta(k) &= \frac{(k-1)k(k+1)^2}{1+2k} \frac{1}{b(k)c(k) - a(k)d(k)} b(k)
\end{aligned} \tag{91}$$

Consider $k \to k+1$ in (90) to get

$$x(k+2) = \alpha(k+1)x(k+1) + \beta(k+1)y(k+1) \tag{92}$$

Invert (90)

$$x(k) = \frac{1}{\alpha(k)\delta(k) - \beta(k)\gamma(k)}(\delta(k)x(k+1) - \beta(k)y(k+1)) \tag{93}$$

and use (93) to eliminate y in (92). One obtains a recursion for x

$$\beta(k)x(k+2) - (\beta(k)\alpha(k+1) + \beta(k+1)\delta(k))x(k+1) + (\alpha(k)\delta(k) - \beta(k)\gamma(k))\beta(k+1)x(k) = 0 \tag{94}$$

that is to say, using (91)

$$k^4 x(k-1) - (2k-1)(2k+1)(2+5k+5k^2)x(k) + 16(1+k)^2 x(k+1) = 0 \tag{95}$$

When multiplied by $16k^2$ the recursion (95) is (42) for $\tilde{x}(k) = 8^k(2k)!k!x(k)$.

In the $\kappa = 5$ case, the iteration (16) for $x(k) = I^{(5)}_{2k,0}$, $y(k) = I^{(5)}_{2k,2}$ and $z(k) = I^{(5)}_{2k,4}$ is

$$\begin{pmatrix} x(k) \\ y(k) \\ z(k) \end{pmatrix} = \begin{pmatrix} \frac{5(1+2k)}{2(k+1)} & \frac{10(1+2k)}{(k+1)} & 0 \\ \frac{1+2k}{k} & \frac{(1+2k)(9+17k)}{2k^2} & \frac{3(1+k)(1+2k)}{k^2} \\ 0 & \frac{6(1+k)(1+2k)}{k(k-1)} & \frac{(1+k)(1+2k)(-8+13k)}{2k(k-1)^2} \end{pmatrix} \begin{pmatrix} x(k+1) \\ y(k+1) \\ z(k+1) \end{pmatrix} \tag{96}$$

Again one may start from the inverse iteration

$$\begin{pmatrix} x(k+1) \\ y(k+1) \\ z(k+1) \end{pmatrix} = \begin{pmatrix} \alpha(k) & \beta(k) & \gamma(k) \\ \delta(k) & \epsilon(k) & \zeta(k) \\ \eta(k) & \theta(k) & \iota(k) \end{pmatrix} \begin{pmatrix} x(k) \\ y(k) \\ z(k) \end{pmatrix} \tag{97}$$

where in particular $\beta(k)\zeta(k) - \gamma(k)\epsilon(k) = 0$ and $\delta(k)\theta(k) - \epsilon(k)\eta(k) = 0$ to account for the vanishing elements of (96). In (97) consider $k \to k+1$

$$z(k+2) = \eta(k+1)x(k+1) + \theta(k+1)y(k+1) + \iota(k+1)z(k+1) \tag{98}$$

$$y(k+2) = \delta(k+1)x(k+1) + \epsilon(k+1)y(k+1) + \zeta(k+1)z(k+1) \tag{99}$$

and $k \to k+2$

$$y(k+3) = \delta(k+2)x(k+2) + \epsilon(k+2)y(k+2) + \zeta(k+2)z(k+2) \tag{100}$$

(99, 100) allow to eliminate the z's in (98)

$$\begin{aligned} &\frac{1}{\zeta(k+2)}(y(k+3) - \delta(k+2)x(k+2) - \epsilon(k+2)y(k+2)) \\ &= \eta(k+1)x(k+1) + \theta(k+1)y(k+1) + \frac{\iota(k+1)}{\zeta(k+1)}(y(k+2) - \delta(k+1)x(k+1) - \epsilon(k+1)y(k+1)) \end{aligned} \tag{101}$$

Invert (97)

$$x(k) = \frac{(-\zeta(k)\theta(k) + \epsilon(k)\iota(k))\,x(k+1) + (\gamma(k)\theta(k) - \beta(k)\iota(k))\,y(k+1)}{\gamma(k)\delta(k)\theta(k) - \alpha(k)\zeta(k)\theta(k) - \beta(k)\delta(k)\iota(k) + \alpha(k)\epsilon(k)\iota(k)} \tag{102}$$

and use (102) to eliminate the y's in (101). One finally obtains a recursion for x, which reads, using the matrix elements in (96)

$$8k^5x(k-1)-(-1+2k)\Big(4\left(5+28k+63k^2+70k^3+35k^4\right)x(k)$$
$$-(1+k)(1+2k)\left(2\left(285+518k+259k^2\right)x(k+1)-225(2+k)(3+2k)x(k+2)\right)\Big)=0 \tag{103}$$

Generalizing this construction to higher order recursions for $\kappa>5$ is straightforward -see [11] for recursions of this type (see also [12] for an example involving $\zeta(5)$ and $\zeta(3)$).

APPENDIX C: A FAMILY OF DOUBLE NESTED INTEGRALS

Coming back to (2,9) use

$$\int_0^\infty g(u)\mathrm{d}u\int_u^\infty f(x)\mathrm{d}x=\int_0^\infty f(u)\mathrm{d}u\int_0^u g(x)\mathrm{d}x$$

to rewrite $I_{\rho^2\alpha^6}$ as

$$\begin{aligned}
I_{\rho^2\alpha^6} = 8&\int_0^\infty \mathrm{d}u\, u\, K_0(u)^2(uK_1(u))^2\int_0^u \mathrm{d}x\, xK_1(x)I_1(x)K_0(x)^2\\
-4&\int_0^\infty \mathrm{d}u\, u\, K_0(u)^2K_1(u)^2\int_0^u \mathrm{d}x\, xK_0(x)xK_1(x)\big(xK_1(x)I_0(x)-xK_0(x)I_1(x)\big)\\
+&\int_0^\infty u\, K_0(u)^4(uK_1(u))^2\mathrm{d}u
\end{aligned} \tag{104}$$

Next use $xI_1(x)K_0(x)+xI_0(x)K_1(x)=1$ so that

$$\begin{aligned}
I_{\rho^2\alpha^6} = 8&\int_0^\infty \mathrm{d}u\, u^3\, K_0(u)^2K_1(u)^2\int_0^u \mathrm{d}x\, xK_1(x)I_1(x)K_0(x)^2\\
+8&\int_0^\infty \mathrm{d}u\, u\, K_0(u)^2K_1(u)^2\int_0^u \mathrm{d}x\, x^3K_1(x)I_1(x)K_0(x)^2\\
-4&\int_0^\infty \mathrm{d}u\, u\, K_0(u)^2K_1(u)^2\int_0^u \mathrm{d}x\, x^2K_0(x)K_1(x)\\
+&\int_0^\infty u\, K_0(u)^4(uK_1(u))^2\mathrm{d}u
\end{aligned} \tag{105}$$

One has for the next to last term in (105)

$$
\begin{aligned}
-4\int_0^\infty \mathrm{d}u\, u\, K_0(u)^2K_1(u)^2\int_0^u \mathrm{d}x\, x^2K_0(x)K_1(x) &= 2\int_0^\infty u\, K_0(u)^2K_1(u)^2\big((uK_1(u))^2-1\big)\mathrm{d}u \\
&= -\frac{2}{3}\int_0^\infty u\, K_1(u)\,\big((uK_1(u))^2-1\big)\mathrm{d}K_0(u)^3 \\
&= \frac{2}{3}\int_0^\infty K_0(u)^3\, \mathrm{d}(uK_1(u))\big((uK_1(u))^2-1\big) \\
&= \frac{2}{3}\int_0^\infty uK_0(u)^4\mathrm{d}u\, - 2\int_0^\infty u\, K_0(u)^4(uK_1(u))^2\mathrm{d}u
\end{aligned} \tag{106}
$$

so that

$$
\begin{aligned}
I_{\rho^2\alpha^6} = &\, 8\int_0^\infty \mathrm{d}u\, u^3\, K_0(u)^2K_1(u)^2\int_0^u \mathrm{d}x\, xK_1(x)I_1(x)K_0(x)^2 \\
&+8\int_0^\infty \mathrm{d}u\, u\, K_0(u)^2K_1(u)^2\int_0^u \mathrm{d}x\, x^3K_1(x)I_1(x)K_0(x)^2 \\
&+\frac{2}{3}\int_0^\infty uK_0(u)^4\mathrm{d}u \\
&-\int_0^\infty u\, K_0(u)^4(uK_1(u))^2\mathrm{d}u
\end{aligned} \tag{107}
$$

Defining

$$
f_n(x) = x^nK_0(x)^2K_1(x)^2 \quad \text{and} \quad g_n(x) = x^nK_0(x)^2K_1(x)I_1(x)
$$

equations (4, 8, 107) imply that

$$
\begin{aligned}
\tilde\zeta(f_3,g_1)+\tilde\zeta(f_1,g_3) =_{PSLQ} &\frac{1}{48}\int_0^\infty u\, K_0(u)^6\,\mathrm{d}u - \frac{3}{160}\int_0^\infty u^3K_0(u)^6\,\mathrm{d}u \\
&-\frac{7}{96}\,\zeta(3) - \frac{31}{1280}\,\zeta(5)
\end{aligned} \tag{108}
$$

as confirmed by a direct PSLQ check.

So the meaning of (9) when rewritten as (108) might be that it is the symmetric form $\tilde\zeta(f_3,g_1)+\tilde\zeta(f_1,g_3)$ which is a linear combination with rational coefficients of simple Bessel integrals of weigth 6 and ζ numbers like $\zeta(5)$ and below. This suggests to look at other symmetric sums of double nested integrals sharing this property. A PSLQ search confirms that $\tilde\zeta(f_5,g_1)+\tilde\zeta(f_1,g_5)$ belongs indeed to this category

$$
\begin{aligned}
\tilde\zeta(f_5,g_1)+\tilde\zeta(f_1,g_5) =_{PSLQ} &\frac{211}{11520}\int_0^\infty u\, K_0(u)^6\,\mathrm{d}u + \frac{3953}{23040}\int_0^\infty u^3K_0(u)^6\,\mathrm{d}u \\
&+\frac{11}{9216} - \frac{1}{9}\,\zeta(3) - \frac{93}{5120}\,\zeta(5)
\end{aligned} \tag{109}
$$

as well as $\tilde\zeta(f_7,g_1)+\tilde\zeta(f_1,g_7)$

$$
\begin{aligned}
\tilde\zeta(f_7,g_1)+\tilde\zeta(f_1,g_7) =_{PSLQ} &\frac{108731}{1728000}\int_0^\infty u\, K_0(u)^6\,\mathrm{d}u + \frac{4256617}{3456000}\int_0^\infty u^3K_0(u)^6\,\mathrm{d}u \\
&+\frac{27877}{460800} - \frac{8}{15}\,\zeta(3) - \frac{279}{5120}\,\zeta(5)
\end{aligned} \tag{110}
$$

and $\tilde{\zeta}(f_3, g_5) + \tilde{\zeta}(f_5, g_3)$

$$\tilde{\zeta}(f_3, g_5) + \tilde{\zeta}(f_5, g_3) =_{PSLQ} -\frac{28921}{691200}\int_0^\infty u\, K_0(u)^6\, \mathrm{d}u + \frac{1151533}{1382400}\int_0^\infty u^3 K_0(u)^6\, \mathrm{d}u + \frac{14653}{184320} + \frac{25}{192}\,\zeta(3) + \frac{279}{20480}\,\zeta(5) \qquad (111)$$

We infer that for any two odd positive integers n, m one should have that $\tilde{\zeta}(f_n, g_m)+$ $\tilde{\zeta}(f_m, g_n)$ can be rewritten as a linear combination of 1, $\int_0^\infty u\, K_0(u)^6\, \mathrm{d}u$, $\int_0^\infty u^3\, K_0(u)^6\, \mathrm{d}u$ and $\zeta(3)$ and $\zeta(5)$. Finding the coefficients of the linear combination can be in principle obtained by generalizing integration by part procedures in [1, 2, 4, 5] to double nested integrals.

References

[1] J. Desbois, C. Furtlehner and S. Ouvry, Random Magnetic Impurities and the Landau Problem, Nuclear Physics B[FS] **453** (1995) 759.

[2] C. Furtlehner and S. Ouvry, Integrals involving four Macdonald functions and their relation to $7\zeta(3)/2$, arXiv:math-ph/0306004.

[3] C. Furtlehner and S. Ouvry, private communication.

[4] S. Ouvry, Random Aharonov-Bohm vortices and some exact families of integrals, JSTAT: Theory and Experiment (2005) P09004.

[5] S. Mashkevich and S. Ouvry, Random Aharonov-Bohm vortices and some exact families of integrals: Part II, JSTAT: Theory and Experiment (2008) P03018.

[6] http://mathworld.wolfram.com/PSLQAlgorithm.html, and references therein.

[7] as far as Apéry's work is concerned see R. Apéry, Irrationalité de $\zeta(2)$ et $\zeta(3)$, Astérisque, 61:11-13, 1979 and the review A. van der Poorten, "A proof that Euler missed...Apéry's proof of the irrationality of $\zeta(3)$, An informal Report" Math. Intelligencer, 1:195-203, 1979.

[8] see for example M. Konsevitch and D. Zagier, Periods (2001) IHES/M/01/22

[9] D. Zagier, Integrals solutions of Apéry-like recurrence equations, in: Groups and Symmetries: From Neolithic Scots to John McKay, CRM Proc. Lecture Notes 47 (Amer. Math. Soc., Providence, RI, 2009), 349-366.

[10] (70) and (72) are listed in G. Almkvist and W. Zudilin, Differential equations, mirror maps and zeta values. In Mirror symmetry. V, AMS/IP Stud. Adv. Math. 38, Amer. Math. Soc. Providence, RI (22006), 481-515.

[11] J. M. Borwein and B. Salvy, A proof of recursion for Bessel moments (2007).

[12] W. Zudilin, A third order Apéry-like recursion for $\zeta(5)$, arXiv:math/0206178.